PMM U.S.S.R.,Vol.54,No.2,pp.258-266,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00


# SCATTERING OF A BENDING WAVE BY A FINITE RECTILINEAR CRACK IN AN ELASTIC PLATE*

I.V. ANDRONOV

The displacement field scattered by a rectilinear thin crack of finite length in a plate vibrating in bending is investigated. The boundary-value problem is reduced to integral equations on a segment by methods analogous to those developed in /1/. These integral equations are later replaced by the method of orthogonal polynomials, by infinite algebraic systems solved by the method of reduction. These systems also enable one to find the asymptotic form of the scattered field in the case of a short crack. The asymptotic form of the radiation pattern of a cylindrical wave diverging from the crack and the effective scattering cross-section are constructed. The results are monitored by using an optical theorem /2/.

1. *Formulation of the problem and change to dimensionless quantities.* The problem of the scattering of an incident field $\xi^{(0)}(x, y)$ by a crack $\Lambda = \{|x| < a, y = 0\}$ in a thin infinitely extended plate consists of seeking the scattered field $\xi^{(1)}(x, y)$, that satisfies the equation of bending vibrations

$$\Delta^2\xi^{(1)} - k_0^4\xi^{(1)} = 0, \quad (x, y) \notin \Lambda \tag{1.1}$$

so that the total field $\xi = \xi^{(0)} + \xi^{(1)}$ satisfies the boundary conditions

$$\begin{gathered} S_2^{\pm}\xi \equiv \lim_{y\to\pm 0} (\xi_{yy} + \sigma\xi_{xx}) = 0 \\ S_3^{\pm}\xi \equiv \lim_{y\to\pm 0} (\xi_{yyy} + (2-\sigma)\xi_{xxy}) = 0, \quad |x| < a \end{gathered} \tag{1.2}$$

Here $k_0$ is the wave number of the plate bending vibrations, and $\sigma$ is Poisson's ratio.

Conditions (1.2), denoting the absence of a bending moment and a transverse force at the crack edges, can be conveniently rewritten in the form

$$(S_n^+ - S_n^-)\xi^{(1)} = 0; \quad S_n^+\xi = 0, \quad |x| < a \tag{1.3}$$

The subscript $n$ takes on the values 2 and 3 here and henceforth. The first condition of (1.3) is satified along the whole axis and does not contain the incident field because of its continuity.

The scattered field must satisfy the radiation condition. To select a physically meaningful solution it is also necessary to specify the following behaviour of $\xi^{(1)}$ in the neighbourhood of the crack tips

$$\begin{gathered} \xi^{(1)} = \xi_0^{\pm} + \xi_1^{\pm}(\theta) r + \xi_{3/2}^{\pm}(\theta) r^{3/2} + \ldots \\ \theta = \operatorname{arctg}\left(\frac{y}{x \pm a}\right), \quad r = ((x \pm a)^2 + y^2)^{1/2} \to 0 \end{gathered} \tag{1.4}$$

The absence of a term of the form $\xi_{1/2}^{\pm}(\theta)r^{1/2}$ follows from the requirement that the energy stored in any bounded domain of the plate must be finite, while the presence of the term $\xi_{3/2}^{\pm}(\theta)r^{3/2}$ results in the singularity $r^{-3/2}$ mentioned in /1/ for the transverse force. It can be shown that despite such a strong singularity of the force, the total energy flux through a circle of small radius enclosing the crack tip will vanish because of the special dependence of the solution on the angle in the limit. This enables us to prove a theorem on the uniqueness of the solution of the boundary-value problem /3/.

We introduce the dimensionless coordinates $x', y'$ and the wave number $k_0'$

$$x' = x/a, \quad y' = y/a, \quad k_0' = k_0'a \tag{1.5}$$

Since the subsequent discussion will make use of dimensionless quantities, the prime will be omitted below. We will return to dimensional quantities only in the final formulas, as will be indicated.

*Prikl.Matem.Mekhan.,54,2,312-321,1990

2. *Reduction of the problem to integral equations of the first kind.* We will seek the scattered field $\xi^{(1)}$ in the form of an expansion in plane waves with the unknown functions $p_j(\tau)$ * (*Kouzov D.P., Boundary-contact problems of the acoustics of a plate-fluid system. Doctorate Dissertation, Acoustics Institute, USSR Academy of Sciences, 1986).

$$\xi^{(1)}=\sum_{j=0}^{3}s_j\int_{-\infty}^{+\infty}\exp(ik_0x\tau)\,p_j(\tau)\,(\tau_-^{j-1}\exp(ik_0|y|\tau_-)-(i\tau_+)^{j-1}\exp(-k_0|y|\tau_+))\,d\tau;\quad s_0=s_2=1,\quad s_1=s_3=\operatorname{sign}(y)\tag{2.1}$$

$\mu_\pm=(1\pm\mu^2)^{1/2}$, the symbol $\mu$ is replaced by different variables $x, t, \tau$. The contour of integration passes along the real axis, bypassing the poles of the integrand at the points $\tau=-1$ and $\tau=1$ from above and below, respectively. The number of components in (2.1) is determined by the order of the differential operator (1.1). The components with numbers 0 and 2 form the part of the field even in $y$, and those with numbers 1 and 3 the part of the field odd in $y$. The behaviour of the functions $p_j$ at infinity is governed by the asymptotic forms (1.4)

$$p_j(\tau)=O(\tau^{1/2-j})\tag{2.2}$$

Therefore, the integrals converge for all values of $x$, $y$.

The representation (2.1) automatically satisfies (1.1) outside the axis $y=0$ since

$$(\Delta^2-k_0^4)\,\xi^{(1)}=4\sum_{j=0}^{3}k_0^{3-j}\int_{-\infty}^{\infty}\exp(ik_0x\tau)\,p_j(\tau)\,d\tau\left(-i\frac{d}{dy}\right)^j\delta(y)$$

In order for (1.1) to be satisfied everywhere outside the crack, it remains to require that the following integrals vanish:

$$\int_{-\infty}^{+\infty}\exp(ik_0x\tau)\,p_j(\tau)\,d\tau=0,\ |x|<1,\quad 0\leqslant j\leqslant 3\tag{2.3}$$

We now turn to the boundary conditions (1.3). Substituting the representation (2.1) and carrying out formal differentiation under the integral sign (we will understand the divergent Fourier integral in the sense of generalized functions), we find from the first condition in (1.3)

$$p_0(\tau)=\sigma\tau^2p_2(\tau),\quad p_1(\tau)=(2-\sigma)\,\tau^2p_3(\tau)\tag{2.4}$$

Taking (2.4) into account, we write the equations originating from the second condition in (1.3) in the form

$$k_0^{n-2}\int_{-\infty}^{+\infty}\exp(ik_0x\tau)\,G_n(\tau)\,p_n(\tau)\,d\tau=i^{n-1}k_0^{-2}S_n\xi^{(0)}(x),\quad |x|<1\tag{2.5}$$

$$G_2(\tau)=i\,[(1-\sigma)\,\tau^2-1]^2\,\tau_-^{-1}-[(1-\sigma)\,\tau^2+1]^2\,\tau_+^{-1}$$
$$G_3(\tau)=-i\,[(1-\sigma)\,\tau^2+1]^2\,\tau_--[(1-\sigma)\,\tau^2-1]^2\,\tau_+$$

The symbol ± is omitted for the operators $S_n$ because of the continuity of the incident field $\xi^{(0)}$.

Therefore, the problem reduces to solving the integral Eqs.(2.3) and (2.5) for $p_2$ and $p_3$. Equations of this type often occur in diffraction problems and are solved by the Wiener-Hopf method for large values of $k_0$, which enables the field to be represented in the form of the superposition of waves multiply rereflected from the ends of the inhomogeneity /4/. We will be interested in not too large $k_0$ and will proceed in a different manner /1/. In order to satisfy the uniform equations outside the crack, we will represent the functions $p_i$ by Fourier transforms in the segment

$$p_n(\tau)=\int_{-1}^{1}\varphi_n(t)\exp(-ik_0t\tau)\,dt\tag{2.6}$$

Substituting these representations into (2.5), and evaluating the integrals with respect to $\tau$, we obtain integral equations of the first kind

$$\mathbf{K}_n\varphi_n\equiv\int_{-1}^{1}K_n(x-t)\,\varphi_n(t)\,dt=i^{n-1}k_0^{-2}S_n\xi^{(0)}(x)\tag{2.7}$$

$$K_n(\rho) = k_0^{n-2} \int_{-\infty}^{\infty} \exp(ik_0\rho\tau) G_n(\tau)\, d\tau$$

The explicit form of the kernel $K_n$ that is a certain combination of Hankel and Macdonald functions, is not required.

3. ***Investigation of the kernels, and the existence and uniqueness of the solutions.*** We extract the highest singularity of the kernels $K_2$ and $K_3$. To do this we indicate the nature of the growth of their Fourier transforms as $\tau \to \infty$

$$G_2(\tau) = -\varkappa \mid \tau \mid + G_2'(\tau), \quad G_2'(\tau) = O(\tau^{-3})$$
$$G_3(\tau) = \varkappa \mid \tau \mid^3 + G_3'(\tau), \quad G_3'(\tau) = O(\tau^{-1}); \varkappa = (1-\sigma)/(3+\sigma)$$

We hence find

$$K_2(\rho) = -2\varkappa k_0^{-2} (d^2/d\rho^2) \ln \mid \rho \mid + K_2'(\rho)$$
$$K_3(\rho) = -2\varkappa k_0^{-3} (d^4/d\rho^4) \ln \mid \rho \mid + K_3'(\rho)$$

The corrections $K_n'$ are functions that are integrable in the segment. Therefore, when (2.7) are next written, the integrals are understood in the ordinary sense:

$$-2\varkappa \frac{d^2}{dx^2} \int \ln|x-t|\, \varphi_2(t)\, dt + k_0^2 \int K_2'(x-t)\, \varphi_2(t)\, dt = iS_2\xi^{(0)}(x) \tag{3.1}$$
$$-2\varkappa \frac{d^4}{dx^4} \int \ln|x-t|\, \varphi_3(t)\, dt + k_0^3 \int K_3'(x-t)\, \varphi_3(t)\, dt = -k_0 S_3\xi^{(0)}(x)$$

Here and henceforth, unless otherwise specified, the integration will be performed between the limits $-1$ and $+1$.

Conditions (2.2) dictate the following behaviour for the function $\varphi_n$

$$\varphi_2(t) = t_-\varphi_2'(t), \quad \varphi_3(t) = t_-^3\varphi_3'(t) \tag{3.2}$$

We will investigate the question of the existence and uniqueness of the solutions of the integral Eqs.(3.1 in the classes of functions with behaviour given by (3.2). Acting with the integration operator two and four times, respectively, on (3.1), we reduce them to the form of integral equations with a logarithmic singularity in the kernels $L_n$

$$L_n\varphi_n \equiv \int \{\ln \mid x-t \mid + L_n'(x-t)\}\, \varphi_n(t)\, dt = h_n(x) \tag{3.3}$$

The right-hand sides $h_2$ and $h_3$ contain two and four arbitrary constants (constants of integration), respectively. As is well-known /5/, the solution of (3.3) exists and is unique for any functions $h_n$ with derivative from the Hölder class and representable in the form

$$\varphi_n(t) = \Phi_n(t)/t_- \tag{3.4}$$

where $\Phi_n(t)$ are functions from the Hölder class.

Using the representation of the kernels $L_n$ in terms of certain analytic functions $a_n(\rho)$ and $b_n(\rho)$

$$L_n(\rho) = \ln \mid \rho \mid + a_n(\rho) \ln \mid \rho \mid + b_n(\rho), \quad a_n(0) = 0 \tag{3.5}$$

it can be shown that for a right-hand side from $C^\infty[-1, 1]$ the functions are $\Phi_n \in C^\infty \times [-1, 1]$ (the result is similar to that presented in /6/, where integral equations with a simpler kernel (3.5) were considered for $a_n(\rho) \equiv 0$). We will first prove an auxiliary lemma.

*Lemma 1.* For any $h$ from $C^{M+2}[-1, 1]$, the solution of the integral equation

$$\int \ln|x-t| \sum_{l=0}^{N} q_l (x-t)^l\, \varphi(t)\, dt = h(x)$$

for any finite $N$ can be represented in the form

$$\varphi(t) = \Phi(t)/t_- \tag{3.6}$$

with the functions $\Phi(t)$ from the class $C^M[-1, 1]$

*Proof.* Using the formula for differentiating expressions of the form $\rho^l \ln|\rho|$, we rewrite the kernel as follows:

$$\sum_{l=0}^{N} q_l' \frac{d^{N-l}}{dx^{N-l}}(\ln|x-t|(x-t)^N) + R(x-t)$$

where $R$ is a certain polynomial. Changing the order of differentiation and integration with respect to $\psi = \mathrm{L}^N\varphi \equiv \int \ln|x-t|(x-t)^N\varphi(t)\,dt$, we obtain the differential equation

$$\sum_{l=0}^{N} q_l' \frac{d^{N-l}}{dx^{N-l}}\psi = h - \int R(x-t)\varphi(t)\,dt$$

Since the right-hand side lies in $C^{M+2}[-1,1]$. the solution of the differential equation is a function from $C^{N+M+2}[-1,1]$ and $d^N\psi/dx^N \in C^{M+2}[-1,1]$. We now understand the expression $d^N\mathrm{L}^N\varphi/dx^N$ as the result of the operator $\mathrm{L}^0 \equiv d^N\mathrm{L}^N/dx^N$ with the kernel $\ln|x-t| + \mathrm{const}$ acting on the function $\varphi$. Therefore, the function $\varphi$ is a solution of the inegral equation $\mathrm{L}^0\varphi = d^N\psi/dx^N$ with right-hand side from $C^{M+2}[-1,1]$. According to /6/ (Theorem 23.2), the solution of such an equation can be represented in the form (3.6) with function $\Phi$ from $C^M[-1,1]$.

We will now prove the property of smoothness of $\Phi_n$ for (3.3).

*Theorem 1.* The solution of the integral Eq.(3.3) can be represented in the form (3.4) with the functions $\Phi_n \in C^\infty[-1,1]$ for any right-hand side from $C^\infty[-1,1]$.

*Proof.* We assume the opposite: let the $N$-th derivative of the functions $\Phi_n$ be discontinuous. Then we expand $a_n$ in a Taylor series and retain $N+1$ terms on the left-hand side and transfer the remainder over to the right-hand side together with a convolution of $\varphi_n$ with $b_n$. Differentiating under the integral sign, it can be shown that the right-hand side is a function from $C^{N+2}[-1,1]$. But according to Lemma 1 $\Phi_n \in C^N[-1,1]$. Therefore, we have arrived at a contradiction, that indeed proves the theorem because of the uniqueness of the solution.

Taking account of the smoothness properties proved for $\Phi_n$ the conditions (3.2) mean that the equalities

$$\Phi_2(\pm1) = 0, \quad \Phi_3(\pm1) = 0, \quad d\Phi_3(\pm1)/dt = 0 \tag{3.7}$$

should be satisfied.

To satisfy these requirements we use the arbitrariness in $h_n$

$$h_2 = i\,(d/dx)^{-2}S_2\xi^{(0)} + c_0 + c_1x, \quad h_3 = -(d/dx)^{-4}S_3\xi^{(0)} + d_0 + d_1x + d_2x^2 + d_3x^3$$

We will show that (3.7) for the constants $c_l$ and $d_l$ are solvable uniquely.

We will first prove such solvability for the first integral equation ($n = 2$). We introduce the functions $\psi_0$ and $\psi_1$ that are solutions of the equations

$$\int L_2(x-t)\,\psi_l(t)\,t_-^{-1}dt = x^l, \quad l = 0, 1 \tag{3.8}$$

into the consideration.

We note that the symbol of the operator $\mathrm{L}_2$ has a positive imaginary part while the kernel $L_2$ is an even function. The following lemma holds for $\psi_l$.

*Lemma 2.* For integral Eqs.(3.8) with kernel of the form (3.5) and sign-definite projections of the symbol in a certain direction in the complex plane, the values of the functions $\psi_l$ at the ends of the interval of integration are non-zero.

*Proof.* Taking account of the properties of evenness of $\psi_0$ and oddness of $\psi_1$ it is sufficient to prove that their values differ from zero at one end.

We assume the opposite. Then differentiating (3.8) with respect to $x$ and transferring the derivative from the kernel to $\psi_l t_-^{-1}$ (there are no terms outside the integral because of the assumption), we have

$$\int L_2(x-t)\,(d\,[\psi_l(t)\,t_-^{-1}]/dt)\,dt = l, \; l = 0, 1$$

Because of the uniqueness of the solution we obtain that for $l = 0$ the expression in the square brackets is independent of $t$, which results in the identity $\psi_0 \equiv 0$ contradicting (3.8), on the basis of the structure of the solution set up in Theorem 1. For $l = 1$ we obtain

$$\psi_1(t)\,t_-^{-1} = \int_0^t \psi_0(t')\,t_-'^{-1}\,dt'$$

We multiply the integral equations for $\psi_0$ by $\bar{\psi}_0(x)x_-^{-1}$ (the bar is the symbol of the complex conjugate) and we integrate over the segment. After changing to the Fourier transform $H$ of the kernel we have

$$\int_0^\infty H(\tau)|u(\tau)|^2 d\tau = \frac{1}{2}\int \psi_0(t)t_-^{-1}dt = \lim_{t\to 1}\psi_1(t)t_-^{-1} = 0$$

$$u(\tau) = \int \psi_0(x)x_-^{-1}e^{i\tau x}dx$$

Because of the sign-definiteness of the projection of the symbol $H(\tau)$ we necessarily arrive at the identity $u(\tau)\equiv 0$, i.e., $\psi_0(x)\equiv 0$, which contradicts (3.8).

On the basis of Lemma 2 and the properties of evenness and oddness of $\psi_l$ it can be seen that the determinant of the matrix of the system of linear algebraic equations is non-zero for the constants $c_0, c_1$. Therefore, the following theorem holds.

*Theorem 2.* The integral Eq.(3.1) for $\varphi_2$ is solvable uniquely in the class of functions of the form (3.2) for any smooth right-hand side.

Consider the integral Eq.(3.1) for $\varphi_3$. We will determine the constants $d_l$ in two stages. We first consider $d_2$ and $d_3$ to be known, while we find $d_0$ and $d_1$ from the system $\Phi_3(\pm 1) = 0$, analogous to that examined above, and therefore, solvable uniquely. Then the solution of the second equation in (3.1) can be represented in the form $\varphi_3(t) = \Phi_3'(t)t_-$ with smooth function $\Phi_3'$. The last pair of conditions in (3.7) denotes $\Phi_3'(\pm 1) = 0$ in terms of $\Phi_3'$. The solvability of this system for $d_2$ and $d_3$ is proved in the same way as the solvability of the first pair of relationships in (3.7) and is based on the properties of solvability of the first equation in (3.1) in the class (3.2) established in Theorem 2, that are analogous to the solvability properties of (3.3) in the class (3.4). Therefore, we arrive at the following assertion.

*Theorem 3.* The solution of the integral Eq.(3.1) for $\varphi_3$ can be represented in the form (3.3) with $\varphi_3' \in C^\infty[-1, 1]$ for any right-hand side from $C^\infty \times [-1, 1]$, and the solution in the class mentioned is unique.

4. ***Scheme for the numerical solution of integral equations.*** We will use the method of orthogonal polynomials for the numerical solution of the integral Eqs.(3.1). By selecting apropriate systems of orthogonal polynomials /7/ this method enables us to take into account the nature of the behaviour of the solutions at the ends of the interval of integration. We will seek the solution $\varphi_n$ in the form of the following expansions with the unknown coefficients $\alpha_l$, $\beta_l$:

$$\varphi_2(t) = t_- \sum_{l=0}^\infty \alpha_l U_l(t), \quad \varphi_3(t) = t_-^3 \sum_{l=0}^\infty \beta_l C_l^{(2)}(t) \tag{4.1}$$

where $U_l$ are Chebyshev polynomials of the second kind and $C_l^{(2)}$ are Gegenbauer polynomials. The choice of these polynomials is determined by the fact that they are eigenfunctions of the principal parts of the integral operators

$$\frac{d^2}{dx^2}\int \ln|x-t|\, t_- U_l(t)\,dt = \pi(l+1)U_l(x)$$

$$\frac{d^4}{dx^4}\int \ln|x-t|\, t_-^3 C_l^{(2)}(t)\,dt = \pi(l+1)(l+2)(l+3)C_l^{(2)}(x)$$

After substituting the expansions (4.1) into the appropriate integral equations, we equate coefficients of polynomials of identical numbers and we obtain infinite systems of linear algebraic equations to determine $\alpha_l$, $\beta_l$

$$-\varkappa(l+1)\alpha_l + \pi^{-2}k_0^2 \sum_{m=0}^\infty A_{lm}\alpha_m = \pi^{-2}f_l \tag{4.2}$$

$$\frac{\varkappa}{4}(l+1)^2(l+2)(l+3)^2\beta_l + \pi^{-2}k_0^3 \sum_{m=0}^\infty B_{lm}\beta_m = \pi^{-2}g_l$$

where $f_l$ and $g_l$ are coefficients of the expansion of the right-hand sides

$$f_l = i\int S_2\xi^{(0)}(x)x_-U_l(x)\,dx, \quad g_l = -k_0\int S_3\xi^{(0)}(x)x_-^3C_l^{(2)}(x)\,dx$$

The elements $A_{lm}$ and $B_{lm}$ of the matrices of the systems are expressed by double integrals

$$A_{lm}=\iint K_2'(x-t)\,t\_U_l(t)\,x\_U_m(x)\,dt\,dx \qquad (4.3)$$
$$B_{lm}=\iint K_3'(x-t)\,t\_{}^3C_l^{(2)}(t)\,x\_{}^3C_m^{(2)}(x)\,dt\,dx$$

We note that the elements $A_{lm}, B_{lm}$ with subscripts of different evenness vanish, which corresponds to partition of the field $\xi^{(1)}$ into even and odd parts in $x$. Therefore, each of the systems (4.2) decomposes into two independent systems for the coefficients $\alpha_l$ and $\beta_l$ with even and odd numbers.

In order to avoid double integrals in the representations (4.3), we turn to Fourier transforms of the kernels $K_n'$. The integrals with respect to $x$ and $t$ are afterwards easily expressed in terms of Bessel functions. Consequently, the elements $A_{lm}$ and $B_{lm}$ are written in the form of integrals of functions with a power-law decrease of the order $O(\tau^{-6})$ at infinity, over the semi-axis

$$A_{lm}=-2\pi^2k_0^{-2}(l+1)(m+1)\,i^{l+m}\int_0^\infty G_2'(\tau)\,J_{l+1}(k_0\tau)\,J_{m+1}(-k_0\tau)\,\tau^{-2}\,d\tau \qquad (4.4)$$
$$B_{lm}={}^1/_2\pi^2k_0^{-3}(l+1)(l+2)(l+3)(m+1)(m+2)(m+3)\,i^{l+m}$$
$$\int_0^\infty G_3'(\tau)\,J_{l+2}(k_0\tau)\,J_{m+2}(-k_0\tau)\,\tau^{-4}\,d\tau$$

The convergence of the integrals (4.4) can be accelerated by extracting singular terms of the form $(x-t)^2\ln|x-t|$ and $\ln|x-t|$, respectively, out of the kernels $K_2'$ and $K_3'$. The integrals containing the extracted terms are evaluated analytically and generate a five-diagonal matrix. After extraction of the singular terms, the decrease of the integrands in the representation of the matrix elements of the system (4.2) is accelerated to $O(\tau^{-10})$. By extracting the next singular terms, a power-law decrease can be achieved with arbitrary exponent. The first step in the procedure described is utilized below to represent the elements $B_{lm}$ and enables us to obtain a more exact estimate of the rate of their decrease in subscripts.

We will solve system (4.2) by the method of reduction.

To justify this method we will estimate the behaviour of the elements $A_{lm}$ and $B_{lm}$ in the subscripts. We consider the representation (4.4) for $A_{lm}$. We divide the semi-axis into two parts by the point $\tau=T$. We use the estimate $|J_m(z)|\leqslant(z/2)^m/m!$ in the interval between 0 and $T$ for the Bessel function and the integrability of the kernel $G_2'(\tau)$. In order for the singularities not to occur at zero after taking the Bessel function outside the integral sign, we will first use the recursion relationships

$$J_m(z)+J_{m+2}(z)=2(m+1)\,J_{m+1}(z)/z$$

Consequently, we have the estimate

$$(l+1)(m+1)\left|\int_0^T G_2'(\tau)\,J_{l+1}(k_0\tau)\,J_{m+1}(k_0\tau)\,\tau^{-2}d\tau\right|\leqslant$$
$$C_1k_0^3\,\frac{(k_0T/2)^{l+m}}{l!\,m!}\left(1+\frac{(k_0T/2)^2}{(l+1)(l+2)}\right)\left(1+\frac{(k_0T/2)^2}{(m+1)(m+2)}\right)$$

In the integral of the Bessel function between $T$ and infinity, we estimate the units and take account of the decrease in the expression $|G_2'(\tau)\,\tau^{-2}|\leqslant C_2\tau^{-5}$

$$(l+1)(m+1)\left|\int_T^\infty G_2'(\tau)\,J_{l+1}(k_0\tau)\,J_{m+1}(k_0\tau)\,\tau^{-2}d\tau\right|\leqslant C_2(l+1)(m+1)\,T^{-4}$$

We select the value of $T$ from the condition for the total estimate to be a minimum, from which we have after some rounding-off

$$|A_{lm}|\leqslant C_3\,(k_0/2)^4\,(l+1)\,(m+1)\,((l+1)!\,(m+1)!)^{-4/(l+m)}$$

On the basis of Stirling's formula for the factorial, the reduced estimate means that the elements $A_{lm}$ decrease at the rate $m^{-2}$ near the diagonal and at the rate $m^{-3}$ for $l\ll m$.

An analogous estimate performed directly for the integral representation for $B_{lm}$ is not satisfied. Consequently, we first extract singular terms of the form $C_4\ln|x-t|$ from the kernel $K_3'$, which we expand in Gegenbauer polynomials

$$B_{lm} = B'_{lm} - {}^1/_8\, k_0^{-3}\pi^2 C_4 (\delta_l^{\,m} (\chi_l (l+3)^2 + 4(l+2) + (l+1)^2/(l+4)) -$$
$$4\delta_l^{m+2}(l+1) + \delta_l^{m+1}(l+3)(l-3))$$
$$\gamma_l = \begin{cases} \ln 2/2, & l = 0 \\ 1/l, & l > 0 \end{cases}$$

The decrease in the integrands in the representation of $B'_{lm}$, is accelerated to $O(\tau^{-10})$, as was noted above, which enables us to obtain the following estimate of the behaviour of the corrections in the subscripts:

$$|B'_{lm}| \leqslant C_5 (k_0/2)^6 l^3 m^3 (l! m!)^{-8/(l+m)}$$

i.e., the elements $B_{lm}$ have a weak growth on the order of $m$ on the diagonals but decrease as $m^{-5}$ far from the diagonals.

The infinite system is quasicompletely regular /8/ if for each row of its matrix the sums of the absolute values of the off-diagonal elements are finite after normalization by the diagonal elements and starting with a certain row number $N$ is less than $1-\varepsilon, \varepsilon > 0$. Taking account of the estimates obtained for the behaviour of the elements $A_{lm}$ and $B_{lm}$ in the subscripts, it can be established that the following theorem is satisfied.

*Theorem 4.* The infinite systems (4.2) are quasiregular for any finite $k_0$.

As we know, the method of reduction converges for quasicompletely regular systems to the solution of an infinite system if the finite system of $N$ first equations is solvable. For large values of $k_0$ the number $N$ becomes large and the matrices of the system (4.2) obviously become ill-posed.

5. ***Investigation of the solution. The radiation pattern.*** After the systems (4.2) have been solved, the scattered field is constructed by means of (2.4), (2.6) and (4.1). We simplify the representation obtained for $\xi^{(1)}$. For this, we take account of (2.4) and rewrite (2.1) in the form of the sum of even and odd displacement field component in $y$

$$\xi^{(1)} = -\int_{-\infty}^{\infty} \exp(ik_0 x\tau)\, p_2(\tau) \left(\frac{\zeta_-}{\tau_-} \exp(ik_0|y|\tau_-) - \frac{\zeta_+}{\tau_+} \exp(-k_0|y|\tau_+)\right) + \tag{5.1}$$
$$\operatorname{sign}(y) \int_{-\infty}^{\infty} \exp(ik_0 x\tau)\, p_3(\tau) (\zeta_+ \exp(ik_0|y|\tau_-) - \zeta_- \exp(-k_0|y|\tau_+))\, d\tau$$

$$\xi_\pm = ((1-\sigma)\tau^2 \pm 1)$$

After substituting (4.1) into (2.6) and evaluating the integrals we obtain Bessel function expansions for the functions $p_{2,3}$

$$p_2(\tau) = \pi \sum_{l=0}^{\infty} \alpha_l (-i)^l (l+1) \frac{J_{l+1}(k\tau_0)}{k_0\tau}. \tag{5.2}$$
$$p_3(\tau) = \frac{\pi}{2} \sum_{l=0}^{\infty} \beta_l (-i)^l (l+1)(l+2)(l+3) \frac{J_{l+2}(k_0\tau)}{(k_0\tau)^2}$$

It can be asserted on the basis of the smoothness of the solutions of the integral equations proved in Sect.3 that the coefficients $\alpha_l$ and $\beta_l$ in the expansions (5.2) decrease with number in a superpower manner /9/. Consequently, the summation limits can be replaced by a certain number $N$ with the introduction of as small an error as desired. Then the field is expressed by a single integral of the sum of a certain finite number of components.

The asymptotic form of the scattered field at infinity is of considerable interest. Let the incident field be a plane wave $\xi^{(0)} = \exp(ik_0(x\cos\vartheta_0 + y\sin\vartheta_0))$. We introduce the polar coordinates $x = r\cos\vartheta$, $y = r\sin\vartheta$ and we evaluate the integrals in (5.1) by the stationary-phase method. The second component in each integral makes an exponentially small contribution as $k_0 r \to \infty$, while the first form a diverging cylindrical wave

$$\xi^{(1)} \sim \left(\frac{2\pi}{k_0}\right)^{1/2} \exp\left(ik_0 r - i\frac{\pi}{4}\right) \Psi(\vartheta, \vartheta_0)$$

The radiation pattern $\Psi$ is expressed in terms of $p_2$ and $p_3$ at the point $\tau = \cos\vartheta$

$$\Psi(\vartheta, \vartheta_0) = -p_2(\cos\vartheta)((1-\sigma)\cos^2\vartheta - 1) + \tag{5.3}$$
$$p_3(\cos\vartheta)((1-\sigma)\cos^2\vartheta + 1)\sin\vartheta$$

Formula (5.3) is exact. It is valid for an arbitrary incident field whose characteristics are contained in the functions $p_2$ and $p_3$, determined according to (5.2) by the solutions of systems (4.2).

We will investigate the asymptotic form $\Psi$ as $k_0 \to 0$, i.e., we study scattering by a short crack. We replace the Bessel functions in (5.2) by the highest terms in the Taylor series expansion

$$p_2(\tau) \approx \frac{\pi}{8} \sum_{l=0}^{\infty} \alpha_l (-i)^l (l+1)(k_0\tau)^l$$

$$p_3(\tau) \approx \frac{\pi}{8} \sum_{l=0}^{\infty} \beta_l (-i)^l (l+1)(l+2)(l+3)(k_0\tau)^l$$

In order to determine the order of the coefficients $\alpha_l$ and $\beta_l$ in $k_0$, we will investigate the asymptotic form of the matrices of system (4.2). We obtain the following estimate for the coefficients $B_{lm}$: $B_{lm} = O(k_0)$. A more exact asymptotic form, that can be obtained by investigating the expansion of the kernel $K_2'$ in the neighbourhood of the point $t = x$, is required for $A_{lm}$

$$A_{00} = \frac{i\pi^3}{32}(3\sigma^2 + 2\sigma + 3) + O(k_2^2), \quad A_{lm} = O(k_0^2)$$

The coefficients $f_l$ and $g_l$ are calculated explicitly for the case of plane-wave incidence and have the asymptotic forms

$$f_l = -i^{l+1} k_0^2 \pi \theta_0(\sigma)(l+1) J_{l+1}(k_0 \cos \vartheta_0)/(k_0 \cos \vartheta_0) \sim$$
$$-{}^1/_2 i^{l+1} k_0^2 \pi \theta_0(\sigma)(l+1)(k_0 \cos \vartheta_0)^l$$
$$g_l = {}^1/_2 i^{l+1} k_0^4 \pi \sin \vartheta_0 \theta_0 (2-\sigma)(l+1)(l+2)(l+3) J_{l+2}(k_0 \cos \vartheta_0)/$$
$$(k_0 \cos \vartheta_0)^2 \sim {}^1/_8 i^{l+1} k_0^4 \pi \sin \vartheta_0 \theta_0 (2-\sigma)(l+1)(l+2)(l+3)\cdot$$
$$(k_0 \cos \vartheta_0/2)^l$$
$$\theta_0(\sigma) = (\sin^2 \vartheta_0 + \sigma \cos^2 \vartheta_0)$$

Solving systems (4.2) for the highest terms of the expansions $\varphi_2$ and $\varphi_3$ we obtain the following radiation pattern asymptotic form:

$$\Psi(\vartheta, \vartheta_0) = \nu\varkappa^{-1}\theta(\sigma)\theta_0(\sigma)\left(i - {}^1/_8 \nu\pi\varkappa^{-1}(3\sigma^2 + 2\sigma + 3) + \right. \tag{5.4}$$
$$\left. i\nu \cos\vartheta \cos\vartheta_0 + O(\nu^2)\right) + 2\nu^2\varkappa^{-1}\theta(2-\sigma)\theta_0(2-$$
$$\sigma)\sin\vartheta \sin\vartheta_0 (i + O(\nu))$$
$$\nu = (k_0 a/2)^2, \quad \theta(\sigma) = (\sin^2\vartheta + \sigma\cos^2\vartheta)$$

(we have returned to dimensional quantities in (5.4)).

As we know, the effective scattering cross-section is defined as the ratio between the energy scattered during diffraction by an inhomogeneity, and the energy arriving per unit length of the incident wave front and is expressed in terms of the radiation pattern by two methods /2/

$$\Sigma = \frac{2\pi}{k_0} \int_{-\pi}^{\pi} |\Psi(\vartheta, \vartheta_0)|^2 d\vartheta, \quad \Sigma = -\frac{4\pi}{k_0} \operatorname{Re}(\Psi(\vartheta_0, \vartheta_0)) \tag{5.5}$$

Let us verify the mentioned identity, called the optical theorem. The highest term in the expansion of $\Psi$ is pure imaginary and, therefore, makes no contribution to the second equality in (5.5). Therefore, it is necessary to compare the contribution of this term to the first equality in (5.5) with the contribution of the real component in the next term of the radiation pattern expansion. Carrying out the appropriate calculations, we see that the optical theorem is satisfied in the highest term and the asymptotic form

$$\Sigma = k_0^3 a^4 \frac{\pi}{32} \frac{3\sigma^2 + 2\sigma + 3}{(1-\sigma)^2(3+\sigma)^2}(\sigma\cos^2\vartheta_0 + \sin^2\vartheta_0)^2 + O(k_0^5 a^6)$$

is satisfied for the scattering cross-section.

The author is grateful to B.P. Belinskii for his interest.

REFERENCES

1. POPOV G.YA., Elastic Stress Concentration Around Stamps, Slits, Thin Inclusions and Reinforcements. Nauka, Moscow, 1982.
2. BELINSKII B.P., Optical theorem for wave scattering in an elastic plate. Proceedings of a Scientific Seminar, Leningrad Branch of the Mathematics Institute. Mathematical Problems of the Theory of Wave Propagation, 104, 11, 1981.
3. TIKHONOV A.N. and SAMARSKII A.A., Equations of Mathematical Physics. Gostekhizdat, Moscow, 1953.
4. BELINSKII B.P., Integral equations of stationary problems of shortwave diffraction by obstacles of segment type, Zh. Vychisl. Mat. mat. Fiz., 13, 2, 1973.
5. GAKHOV F.D., Boundary-Value Problems, Nauka, Moscow, 1977.
6. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems of Elasticity Theory, Nauka, Moscow, 1974.
7. ABRAMOWITZ M. and STEGUN I.M.Eds. Handbook on Special Functions with Formulas, Graphs and Mathematical Tables, Nauka, Moscow, 1979.
8. KANTOROVICH L.V. and KRYLOV V.I., Approximate Methods of Higher Analysis. Gostekhizdat, Moscow-Leningrad, 1950.
9. NATANSON I.P., Constructive Theory of Functions. Gostekhteorizdat, Moscow-Leningrad, 1949.

Translated by M.D.F.

*PMM U.S.S.R.*,Vol.54,No.2,pp.266-271,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00


# THE BOUNDARY-LAYER METHOD IN THE FRACTURE MECHANICS OF COMPOSITES OF PERIODIC STRUCTURE*

A.L. KALAMKAROV, B.A. KUDRIAVTSEV and V.Z. PARTON

The problem of a rectilinear crack in a composite material of doubly-periodic structure is considered. It is assumed that the dimensions of the crack are considerably greater than the cell of material periodicity. A boundary-layer method based on the use of the asymptotic method of averaging periodic structures, taking additional solutions of boundary-layer type /1/ into account to allow the edge effect that occurs near the boundary of the crack outline to be considered, is proposed for analysing the stress field in the neighbourhood of a macrocrack.

Analysis of the stress field in highly inhomogeneous (composite) materials with an idealized smooth macrocrack is usually performed by replacing the inhomogeneous composite medium by a certain homogeneous anisotropic medium that is equivalent to the composite material with respect to the average reaction. Such an approach enables the computation of the average stress field in the composite with a macrocrack to be reduced to solving elasticity theory problems for an anisotropic homogeneous material with a mathematical slit. If the material has a periodic structure (as is true of many composites), the average (effective) characteristics of the equivalent should be determined by the method of averaging periodic structures /1-3/ which yields an asymptotically correct approximation to the exact solution of the problem for the initial inhomogeneous medium. The averaging method here allows the local structure of the fields being investigated to be determined with a high degree of accuracy. This approach was used in /4/ to analyse the stress field near a macrocrack in laminar composites of periodic structure. In a number of cases formulas were obtained for

**Prikl.Matem.Mekhan.*,54,2,322-328,1990